THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XVII. FEBRUARY, 1910. NO. 2.

ON THE TEACHING OF THE ELEMENTS OF PLANE TRIGONOMETRY.

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In teaching we proceed from that which is familiar to that which is new. The successful teacher leads the student into a knowledge of Trigonometry by making the successive steps to it as gradual and natural as possible. From his knowledge of Geometry the student has usually a fair understanding of the relations between the sides and angles of a triangle; and of the different kinds of triangles, he is most familiar with the right triangle. Hence it is best to commence with the trigonometric functions of an acute angle and to define them as the ratios of the sides of a right triangle having that acute angle as one of its angles. The functions should be defined and written in pairs as reciprocals of each other in order to aid the memory and to emphasize one of the most important of their functional relations. No attempt should be made at the start to teach the student any particular definition of angle; let angles be to him just what they were to him in his Geometry. The general definition of angle should not be given until the student has mastered the right triangle. Nor is it necessary at this time to introduce circular measure (radians). Only even degrees should be given at first, then minutes or decimal parts of a degree. The use of seconds to to any considerable extent in a first course in Trigonometry has a tendency to obscure the theory, and the additional calculations involved are apt to degenerate into mere drudgery for student and teacher alike. The division of the degree into decimal parts, instead of using minutes and seconds, has much favor by expert computers. Irrespective of what the future of the Metric System may be in the United States, it seems certain that the decimal division of the degree is fast gaining ground in both theoretical and practical work. And right here I wish to record a most emphatic protest against a notation in which $36.2°$ is written $36°.2$. There is no reason whatever for inserting the unit of measurement between the digits of a number. What would we think of the engineer who wrote 127.ft.36 instead of 127.36

ft., and how would it look for a merchant to advertise a marked down sale of straw hats from 2.$50 to 1.$49!

The success or failure of a course in any subject depends in a large measure on a few of the first lessons. This is especially true of Trigonometry. If you begin by giving the average student a general definition of the trigonometric functions, he is apt to become badly confused and easily gets discouraged. On the other hand, if the student in the first few lessons learns the solution of right triangles thoroughly (using the natural values of the functions) he already knows how to use some of the most powerful tools in Trigonometry. In fact, if the student can apply the three following rules, namely:

Side opp. an acute angle=hypot.×sine of the angle,

Side adj. an acute angle=hypot.×cosine of the angle,

Side opp. an acute angle=adj. side×tangent of the angle,

he is already able to solve a large number of the trigonometric problems occurring in the elements of pure and applied mathematics. If we think of the applications of Trigonometry in the elements of Analytic Geometry, for instance, it is surprising how little trigonometric knowledge outside of these three rules is really necessary. When the student has mastered the right triangle and has applied this knowledge to the solution of numerous practical problems he begins at once to realize the utility and beauty of Trigonometry and so at the very start becomes interested and gains a confidence that will stand him in good stead later on.

Logarithms should not be introduced until the student has mastered the fundamental principles of Trigonometry and has used the natural functions in the solution of both right and oblique triangles. If we begin by using logarithms in our calculations the student naturally associates the trigonometric functions with logarithms and after a time he finds it difficult to separate the two notions. It is very important to impress upon the student the fact that in Trigonometry logarithms are employed chiefly for the purpose of minimizing the labor connected with the computations. Beginning with the natural functions also introduces the subject of interpolation as a logical sequence.

After the right triangle has been digested the following general definition of angle should be given:

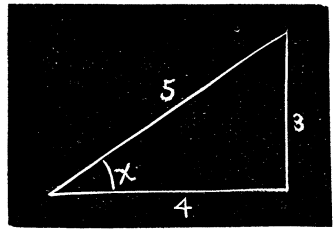
An angle may be considered as generated by a line which first coincides with one side of an angle, then revolves about the vertex, and finally coincides with the other side.

Positive and negative angles and angles of any magnitude should then be defined.

Next comes the definitions of the trigonometric functions of any angle using rectangular coordinates. The student has probably already used rectangular coordinates in his Algebra, so that these definitions will appear perfectly natural to him. If the student is not familiar with coordinates he

should be taught their use before proceeding further. Having given the value of a trigonometric function it is now easy to construct geometrically all the angles which satisfy the given value, and to find the values of the other five functions. Or, we may express any five of the functions in terms of the sixth. If the construction of the angle is not required, the process may be shortened by simply drawing a right triangle to serve as a check on the numerical part (not the algebraic signs) of the work. Thus having given $\sec x=\frac{5}{4}$, to find the value of the other five functions. Here

$$\sec x=\frac{5}{4}=\frac{\text{hypt.}}{\text{adj. leg}}$$



$$\text{opp. leg}=\sqrt{(5^2-4^2)}=3.$$

The numerical values of the functions will then be:

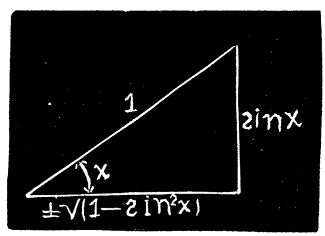
$\sin x=\frac{3}{5}$ $\qquad$ $\csc x=\frac{5}{3}$
$\cos x=\frac{4}{5}$ $\qquad$ $\sec x=\frac{5}{4}$
$\tan x=\frac{3}{4}$ $\qquad$ $\cot x=\frac{4}{3}$

The given secant being positive, the angle will lie either in the first or the fourth quadrants. If the angle lies in the first quadrant all the functions are positive and the above results are correct as they stand. If the angle lies in the fourth quadrant, we merely change the algebraic signs as follows:

$\sin x=-\frac{3}{5}$ $\qquad$ $\csc x=-\frac{5}{3}$
$\cos x=\frac{4}{5}$ $\qquad$ $\sec x=\frac{5}{4}$
$\tan x=-\frac{3}{4}$ $\qquad$ $\cot x=-\frac{4}{3}$

Again, express in terms of $\sin x$, the other five functions of x. Here

$$\sin x=\frac{\sin x}{1}=\frac{\text{opp. leg}}{\text{hyp.}}$$



$\text{adj. leg}=\pm\sqrt{(1-\sin^2 x)}$. Hence

$\sin x=\sin x$ $\qquad$ $\csc x=\dfrac{1}{\sin x}$

$\cos x=\pm\sqrt{(1-\sin^2 x)}$ $\qquad$ $\sec x=\pm\dfrac{1}{\sqrt{(1-\sin^2 x)}}$

$\tan x=\pm\dfrac{\sin x}{\sqrt{(1-\sin^2 x)}}$ $\qquad$ $\cot x=\pm\dfrac{\sqrt{(1-\sin^2 x)}}{\sin x}$

The line definitions (using the unit circle) of the trigonometric functions should now be given. By means of these definitions the limits of the functions, as the angle approaches a multiple of a right triangle, can be illustrated geometrically in a manner most satisfactory to the student. The fundamental relations between the functions (except the reciprocal relations which follow at once from the ratio definitions) are obtained most naturally from the line definitions, as is also the proof of the addition theorem.

Circular measure should now be given with numerous examples illustrating its relations to degree measure. The student is now ready for the proof of

$$\underset{x=0}{\text{limit}}\frac{\sin x}{x}=1 \text{ and } \underset{x=0}{\text{limit}}\frac{\tan x}{x}=1,$$

two limits of great importance in both pure and applied mathematics. Or,

THEOREM. *We may replace $\sin x$ and $\tan x$ in our calculations by x when x is a very small angle and is expressed in circular measure.*

Attention should be called to the fact that when we wish to find the functions of angles near 0° or 90°, ordinary interpolation will in general give inaccurate results, and that the above theorem should be used instead.

In the reduction of functions of any angle to the functions of an acute angle the student should be encouraged to use the forms

$$180^\circ \pm x, \text{ or } 360^\circ \pm x,$$

for then the name of the function remains unchanged throughout the operation and there is less liability of making a mistake. And it should be pointed out that one of the principal reasons for making such a reduction is that our tables give the values of the trigonometric functions of angles from 0° to 90° only.

In a first course the student should not be required to learn the proof of the addition theorem for any but acute angles; but his attention should be called to the fact that the theorem actually does hold true for any angles, and this statement should be illustrated by examples.

Functions of twice an angle and half an angle, in terms of the functions of the angle, should now be given. More prominence than is customary should be given to the formulas

$$\tan\frac{x}{2}=\frac{\sin x}{1+\cos x}=\frac{1-\cos x}{\sin x},$$

for they express $\tan\frac{x}{2}$ rationally in terms of $\sin x$ and $\cos x$.

In proving the four formulas usually commencing with

$$\sin A + \sin B = 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B),$$

we should emphasize the important fact that they express the algebraic sum of sines or cosines in terms of their products.

Care should be taken that the student does not get tied up to any particular set of letters, Greek or Roman, as symbols for angles, or to any particular forms of the formulas. You have probably met the boy who could prove

$$\sin X - \sin Y = 2\cos\tfrac{1}{2}(X+Y)\sin\tfrac{1}{2}(X-Y)$$

all right, but who was completely floored by

$$2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} = \sin\alpha - \sin\beta,$$

or who could not recognize as identical the expressions

$$2\sin^2\frac{x}{2} = 1 - \cos x, \quad \tfrac{1}{2} - \tfrac{1}{2}\cos A = (\sin\tfrac{1}{2}A)^2.$$

A certain degree of uniformity in notation and form is desirable when teaching a beginner, but when trigonometric analysis is reached in the course it is well to make a point of freely varying both notation and form.

Now comes the derivation of the general value for all the angles having the same value of a function, the introduction of inverse trigonometric functions, and the solution of trigonometric equations.

Processes should be summarized into working rules whenever practicable. As for instance, the following:

General directions for solving a trigonometric equation.

First step. If multiple angles, fractional angles, or the sums and differences of angles are involved, reduce all to functions of a single angle, and simplify.

Second step. If the resulting expressions are not readily reducible to the same function, change all the functions into sines and cosines.

Third step. Clear of fractions and radicals.

Fourth step. Change all the functions to a single function.

Fifth step. Solve for the one function now occurring in the equation, and express the general value of the angle thus found. Only such values of the angle which satisfy the given equation are solutions.

Whether or not the graphs of the functions should be given in a first course depends on the time allowed. These graphs illustrate very vividly the property of the periodicity of the trigonometric functions. After a graph has been plotted from the calculated values of a function the student should be taught how to plot the graph by purely geometric methods from the unit circle.

As soon as the law of sines, law of cosines, and law of tangents have been derived, they should be employed in the solution of some oblique triangles making use of the natural values of the functions. Areas of triangles should also be found making use of the natural values.

And now comes the theory and use of logarithms. It is astonishing how incomplete the treatment of these topics is in almost all the current text books. Unless he has the aid of a teacher the student usually runs up against a stone wall when he comes to the application of logarithms to calculations. There are many tricks in the mathematician's trade when it comes to the use of logarithms. The texts do not always explain fully these artifices of the calculator or the peculiarities of the particular tables used. With the teacher this has become second nature and he sometimes becomes impatient at the apparent stupidity of the pupil. It is as if we expected the student to be born with an instinct for the use of logarithms! The teacher should insist on having the calculations set down according to some set form or scheme. In logarithmic computations the student should always write down an outline or skeleton of the computation before using his table at all. For, it saves time to look up all the logarithms at once and, besides, it reduces the liability of error to thus separate the theoretical part of the work from that which is purely mechanical.

All results should be verified by the student himself. The importance of this has been generally overlooked by teachers. A student gains much in interest and self-confidence when he feels independent of both his book and his teacher when it comes to proving the accuracy of his results.

It is imporiant that the student should draw the figures connected with the problems as accurately as possible. This not only leads to a better understanding of the problems themselves, but also gives a clearer insight into the meaning of the trigonometric functions and makes it possible to test roughly the accuracy of the results obtained. The only instruments necessary are a graduated ruler and a protractor, and the student should be advised to use them freely.

Throughout the subject practical problems relating to matters of common knowledge should be given as far as possible. And here, at least, we should call the attention of the student to the fact that in the examples usually given it has been assumed that the given data were exact. That is, if two sides and the inclined angle of a triangle are given, as 135 ft., 217 ft., and 25.3° respectively, we have taken for granted that these numbers are not subject to errors made in measurement. This is in accordance with the

plan followed in the problems that the student has solved in Arithmetic, Algebra, and Geometry. It should not be forgotten, however, that when we apply the principles of Trigonometry to the solution of practical problems,—engineering problems, for instance,—it is usually necessary to use data which have feen found by actual measurement and therefore are subject to error. For instance, if the length of a line is measured by a steel tape, account must be taken of the expansion due to heat as well as the sagging of the tape under various tensions. And in making several measurements one should carefully see that they are made with about the same precision. Thus, it would be folly to measure one side of a triangle with much greater care than another; for, in combining these measurements in a calculation, the result would at best be no more accurate than the worst measurement. Similarly, the angles of a triangle should be measured with the same care as the sides. The number of significant figures in a measurement are supposed to indicate the care that was intended when the measurement was made. In ordinary engineering practice only the first three or four significant figures of the measurement are not subject to error. It is therefore evident that the use of five or six place tables in calculations involving these measurements introduces an unnecessary refinement and merely adds to the labor without making the results more accurate than they would be if four place tables had been used. In a large number of cases three place tables are accurate enough.

At regular intervals the student should be required to write down from memory a list of all the fundamental formulas studied.

Brief notes on the history of the science should be given by the teacher as opportunity offers.

The computation of logarithms and of the trigonometric functions from series, De Moivre's Theorem, and the hyperbolic functions, do not properly belong to a first course in Trigonometry.

FACSIMILE EDITIONS OF JOHN BOLYAI'S SCIENCE ABSOLUTE OF SPACE.

By GEORGE BRUCE HALSTED.

At last, any one has a chance to see just how looked the most extraordinary two dozen pages in the history of human thought.

How John Bolyai himself looked, the world can never know, for before it woke up to his genius he was dead of disgust and so covered with oblivion that no picture of him remains, though his love child Dyonis, son of Rosa Orbân, still lives. His eyes were blue. So much I read in his passport